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Anomaly for Model Building

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Abstract

A simple algorithm to calculate the group theory factor entering in anomalies at four and six dimensions for SU(N) and SO(N) groups in terms of the Casimir invariants of their subgroups is presented. Explicit examples of some of the lower dimensional representations of $SU(n), n \leq 5$ and SO(10) groups are presented, which could be used for model building in four and six dimensions.

The consistency of any gauge theory requires that the sum of anomalies [1] due to all the fermions present in the theory should cancel. The anomaly cancellation is necessary because any classical symmetry is broken by quantum effects in the presence of anomaly. In other words, any gauge theory with non-vanishing anomaly is non-renormalizable [2].

In the standard model of electroweak interactions, the fermion content is just right to make the theory anomaly free. In any extensions of the standard model, particularly the ones which introduces additional gauge symmetry, the most severe constraints come from the anomaly cancellation conditions. Anomaly plays crucial roles in theories of dimension higher than four. Since all representations in odd dimensions contains both left and right chiral fields, there is no anomaly. But in dimensions six, eight or ten, any theory has to cancel higher dimensional anomalies. In higher dimensional theories if the space is compactified in an orbifold, then the orbifold compactification also lead to anomaly at the fixed points. Consistency of such theories then require that the brane anomalies at the fixed points should vanish.

Fermions in the loop contributes to anomaly. So, in non-supersymmetric theories the fermion representations are constrained by anomaly. But in supersymmetric theories, any chiral superfield would contribute to anomalies. So, the superfields containing both the scalar and fermion representations are constrained by the anomaly cancellation requirement.

The cancellation of anomaly is thus an integral part of constructing any consistent model in four or higher dimensions. It is thus important to know the group theory factor of any representation contributing to anomaly in four or higher dimensions. In four dimensions one needs to calculate the triangle anomaly, while at six dimensions it is a box anomaly and at eight dimensions it is pentagon anomaly. These group theory factors for higher groups become difficult to calculate. There are rigorous methods for searching anomanly free theories that are used usually [3]. In this article we present a simple algorithm to calculate the group theory factor appearing in the expression for anomaly in four and six dimensions for Lie groups, which could be useful for model building.

In four dimensions, if the fermions belong to a representation $\mathcal{R}$ of $\mathcal{G}$, then the group factor in the expression for anomaly can be written in terms of the generators $T^a(\mathcal{R}_1)$ for this representation $\mathcal{R}_1$ of $\mathcal{G}$. Then the contributions of fermions in a representation $\mathcal{R}_r$ to anomaly will be proportional to

$$ {}_3\mathcal{A} = \text{tr } [T^a(\mathcal{R}_r) T^b(\mathcal{R}_r) T^c(\mathcal{R}_r)]. \tag{1} $$

We shall use the notation ${}_n\mathcal{A}$ to represent anomalies, so that triangle anomaly in four dimensional theories is represented by $n = 3$; box anomaly in six dimensions with $n = 4$ and pentagon anomaly in eight dimensions by $n = 5$.

In general, $n = d/2 + 1$ for anomaly in a d-dimensional theory and the group factor for fermions or chiral superfields in a representation $\mathcal{R}_r$ entering in the expression for anomaly is given by

$$ {}_n\mathcal{A} = \text{tr}_{\mathcal{R}_r} \, T^n = \text{tr } [T^{a1}(\mathcal{R}_r) T^{a2}(\mathcal{R}_r) \ \cdots \ T^{an}(\mathcal{R}_r)]. \tag{2} $$

In four dimensions this anomaly factor for all the fermions or all the superfields should cancel for consistency. In higher dimensional theories one should apply this factor with caution.

Consider a six dimensional orbifold model [4, 5], compactified on $\mathcal{R}^4 \times T^2/Z_2$. There will be four dimensional anomaly at the fixed points and also the six dimensional anomaly at the bulk. If the gauge group in the bulk is G and at the fixed points only the group H acts, then the only non-vanishing anomaly at the fixed points will be restricted to the subgroup H of G. The brane anomalies are also associated with the parities that acts on the fields due to the action of the discrete Z_2 symmetry. So, the choice of parity to break $N = 2$ supersymmetry to $N = 1$ supersymmetry implies that the spinor in a vector superfield and spinor in a scalar superfield contribute to anomalies with opposite sign. There is another difference in anomalies at six dimensions. In some cases there are contributions of the form

$$ A_{red} = (\text{tr}_{R_r} \, T^2)^2 $$

in addition to the usual $\mathrm{tr}_{R)r}\ T^4$ terms. This factorized contributions, also known as reducible anomalies, does not have any analogy in four dimensions and could be cancelled by introducing antisymmetric tensor fields and utilizing the Green-Schwarz mechanism. We shall thus calculate the group factor for the irreducible anomaly contributions, given by ${}_4\mathcal{A}$. Since there are no independent 4th order invariants for the groups SU(2) and SU(3) and $\mathrm{tr}_{R_r}T^4 = (\mathrm{tr}_{R_r}T^2)^2$, we shall not present ${}_4\mathcal{A}$ for these two groups.

In the present approach all invariants are calculated in terms of invariants of the subgroups. Consider the subgroup

$$\mathcal{G}_1 \times \mathcal{G}_2 \subset \mathcal{G}, \tag{3}$$

where $\mathcal{G}_2 = U(1)$ is an abelian subgroup of $\mathcal{G}$. We can then decompose any representations of $\mathcal{G}$ under $\mathcal{G}_1 \times \mathcal{G}_2$ subgroup as

$$\mathcal{R} = \sum_i (r_i, f_i) \tag{4}$$

where f_i are the $U(1)$ quantum numbers of $\mathcal{G}_2$. For any group $SU(m)$ we can write down the decomposition of the fundamental m dimensional representation under the subgroup $SU(m-1) \times U(1)$ as

$$m = (m-1, 1) + (1, -m+1). \tag{5}$$

The second numbers 1 and $(-m+1)$ are the $U(1)$ quantum numbers in this decomposition. Using the product decomposition formulas we can find out the decomposition of all other representations of $SU(m)$ under the subgroup $SU(m-1) \times U(1)$.

We can then use the formulas

$$\begin{aligned}
{}_3\mathcal{A}(\mathcal{R}) &= \sum_i T^2(r_i) \cdot f_i \\
{}_4\mathcal{A}(\mathcal{R}) &= \sum_i {}_3\mathcal{A}(r_i) \cdot f_i \\
{}_n\mathcal{A}(\mathcal{R}) &= \sum_i {}_{(n-1)}\mathcal{A}(r_i) \cdot f_i
\end{aligned} \tag{6}$$

to calculate the anomalies for the representation $\mathcal{R}$ of the group $\mathcal{G}$ in terms of the invariants of the subgroup $\mathcal{G}_1 \times \mathcal{G}_2$. For verification of the results we also use the formulas

$$ {}_n\mathcal{A}(\mathcal{R}) = \sum_i {}_n\mathcal{A}(r_i). \tag{7} $$

For completeness we also present a couple of useful relations

$$ \begin{aligned} {}_n\mathcal{A}(\mathcal{R}_1 + \mathcal{R}_2) &= {}_n\mathcal{A}(\mathcal{R}_1) + {}_n\mathcal{A}(\mathcal{R}_2) \\ {}_n\mathcal{A}(\mathcal{R}_1 \times \mathcal{R}_2) &= {}_n\mathcal{A}(\mathcal{R}_1) D(\mathcal{R}_2) + {}_n\mathcal{A}(\mathcal{R}_2) D(\mathcal{R}_1). \end{aligned} \tag{8} $$

Thus by writing down the decomposition of any representation under its subgroup containing a $U(1)$ factor, it will be possible to calculate the irreducible group factor entering in the expression for anomalies. For the cancellation of anomalies in any theory, we require only this factor and hence these results will be extremely useful while building models in many extensions of the standard model.

Let us first consider the group SU(2). Since all representations are pseudoreal, all four dimensions triangle anomalies vanishes. There are also no fourth order invariants and hence we have to worry only about the quadratic Casimir invariants. For the group SU(2) the quadratic Casimir invariants are given by

$$ T^2(N) = \sum_{i=-(N-1)/2}^{(N-1)/2} |i|^2 \tag{9} $$

for an N dimensional representation.

We shall next consider the group SU(3). The fourth order invariants are again absent and hence we have to compute the quadratic Casimir invariants and the triangle anomalies for the different representations. Using the decompositions of the representations of SU(3) under $SU(2) \times U(1)$ as

Table 1: Quadratic Casimir invariant and triangle anomalies of SU(3). There are also no 4th order invariants.

$\mathcal{R}_r$	Index (l)	${}_3\mathcal{A}$
3	1	1
$\bar{3}$	1	-1
6	5	7
$\bar{6}$	5	-7
8	6	0
10	15	27
$\bar{10}$	15	-27
15	20	14
$15'$	35	77
21	70	-182
24	50	-64
27	54	0

$$\begin{aligned}
3 &= (2,1) + (1,-2) \\
\bar{3} &= (2,-1) + (1,2) \\
6 &= (3,2) + (1,-4) + (2,-1) \\
& \cdots \quad \cdot\cdot\cdot
\end{aligned}$$

we can use equations 6 and 7 to calculate the quadratic Casimir invariants and the anomalies, which is presented in table 1. The triangle anomalies for SU(3) can be computed using the formula presented in ref. [6].

Proceeding in the similar way, we can calculate the triangle and box anomalies for the groups SU(4) and SU(5), which are presented in tables 2 and 3. The triangle anomalies can again be computed and compared following ref. [6]. Calculating the box anomalies are more involved. In general,

Table 2: Anomalies for the group SU(4).

$\mathcal{R}_r$	Index (l)	${}_3\mathcal{A}$	${}_4\mathcal{A}$
4	1	1	1
$\bar{4}$	1	−1	1
6	2	0	−4
10	6	8	12
$\bar{10}$	6	−8	12
15	8	0	8
20	13	−7	−11
$\bar{20}$	13	7	−11
20′	16	0	−56
20”	21	−35	69
35	56	112	272
36	33	21	57
45	48	48	24
50	70	0	−380

the relations involve both reducible as well as irreducible anomalies. For the group $SU(n)(n > 3)$ the anomaly for the adjoint representation can be written as

$$ {}_4\mathcal{A}(adj) = 2\ n\ {}_4\mathcal{A}(fund) + 6\ (\mathrm{tr}_{(fund)}T^2)^2 \tag{10} $$

in terms of the invariants of the fundamental representations. However for purpose of anomaly cancellation in six dimensional theories we are interested in only the irreducible anomalies and hence the present method will serve the purpose. This procedure can be extended to any higher groups in the same way.

We shall now consider a slightly non-trivial case of the group SO(10). We consider the decomposition of SO(10) under the subgroup $SU(5)\times U(1)$. The vector and the spinor representations of SO(10) decompose under $SU(5) \times$

Table 3: Anomalies for the group SU(5).

$\mathcal{R}_r$	Index (l)	${}_3\mathcal{A}$	${}_4\mathcal{A}$
5	1	1	1
$\bar{5}$	1	-1	1
10	3	1	-3
$\bar{10}$	3	-1	-3
15	7	9	13
$\bar{15}$	7	-9	13
24	10	0	10
35	28	-44	82
$\bar{35}$	28	44	82
40	22	-16	-2
45	24	-6	-6
50	35	-15	-55
70	49	29	79
$70'$	84	-156	354
75	50	0	-70

$U(1)$ as

$$\begin{aligned} 10 &= (5,2) + (\bar{5},-2) \\ 16 &= (1,-5) + (\bar{5},3) + (10,-1). \end{aligned}$$

If we now calculate the triangle anomalies in terms of the triangle anomalies of SU(3) representations, then it is obvious that both the representations 10 and 16 have vanishing anomalies, since the SU(5) anomalies ${}_3\mathcal{A}(5) = {}_3\mathcal{A}(10) = -{}_3\mathcal{A}(\bar{5}) = 1$. It is also well-known that all representations of $SO(10)$ group are anomaly-free. Using equation 6 we can calculate the box anomalies for the representations of SO(10), which are given by

$$\begin{array}{cccccccccc} \mathcal{R} & \rightarrow & 10 & 16 & 45 & 54 & 120 & 126 & 210 \\ {}_4\mathcal{A} & \rightarrow & 4 & -4 & 8 & 72 & -8 & -104 & 120 \end{array}$$

modulo a normalization factor, which will not change the condition for anomaly cancellation. From these the two relations follow:

$$ {}_4\mathcal{A}(45) = 2\ {}_4\mathcal{A}(10) \quad \text{and} \quad {}_4\mathcal{A}(16) = {}_4\mathcal{A}(\bar{16}) = -{}_4\mathcal{A}(10). $$

These relations are true only for the irreducible anomalies, as we discussed earlier. This method can be extended to higher dimensional theories and to all the Lie groups.

In summary, we presented a simple algorithm of calculating anomalies at four and six dimensions for all the Lie groups. We gave explicit example for the groups SU(3), SU(4), SU(5) and SO(10), which are extensively used in building orbifold grand unified theories and Higgsless models in six dimensions and extensions of the standard models.

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